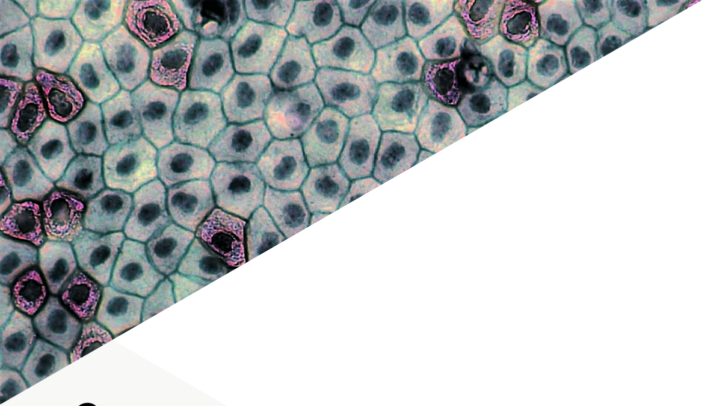
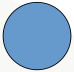


Growth-fragmentation and quasi-stationary methods

Denis Villemonais **Alex Watson**

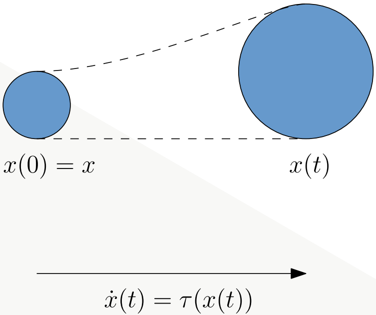
BioHasard, 28 May 2021

A model of growth-fragmentation



$x(0) = x$

A model of growth-fragmentation



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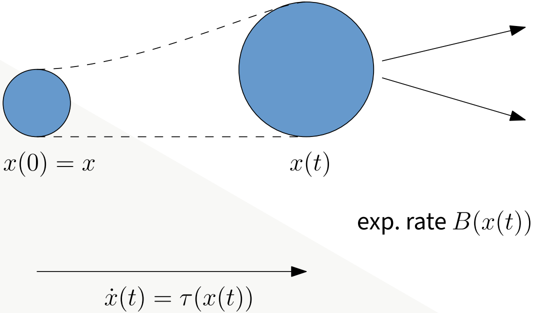

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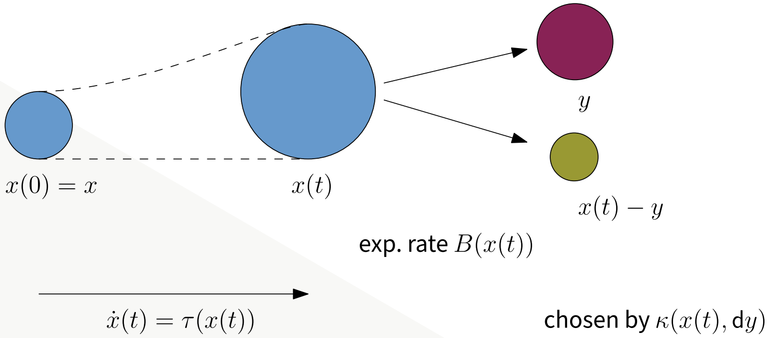

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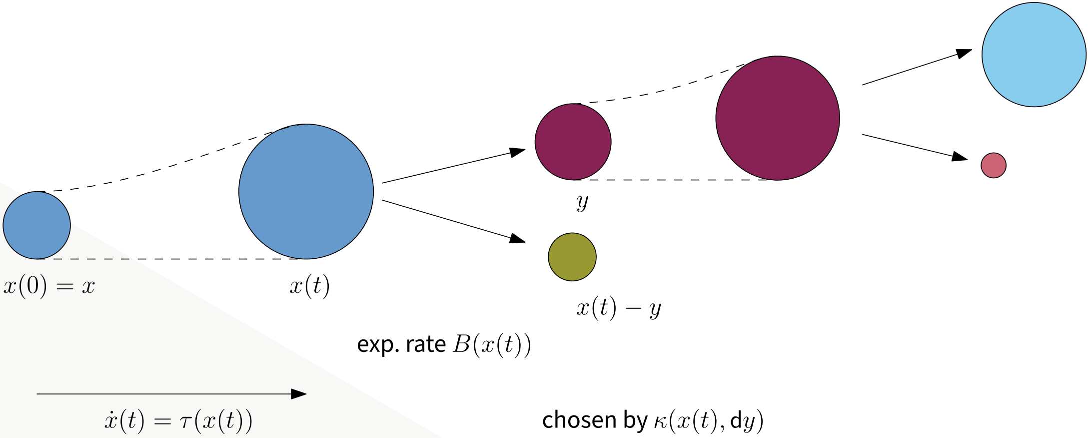

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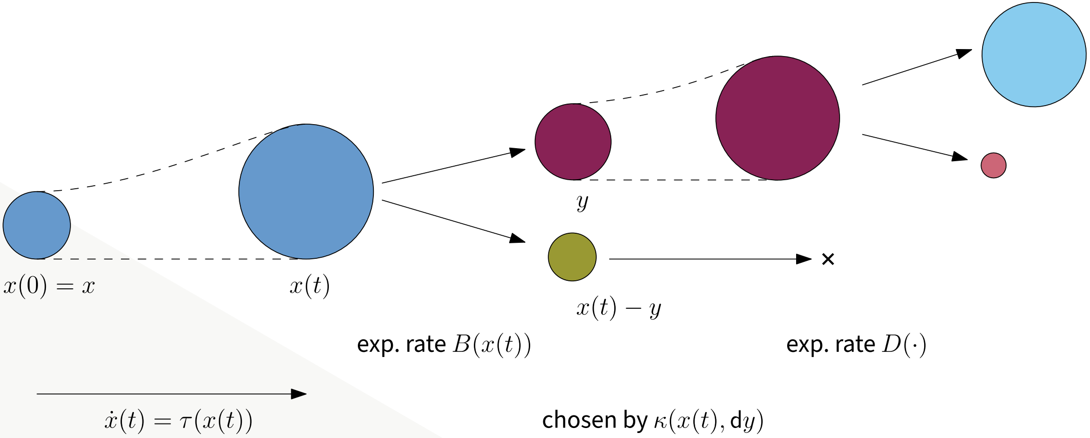

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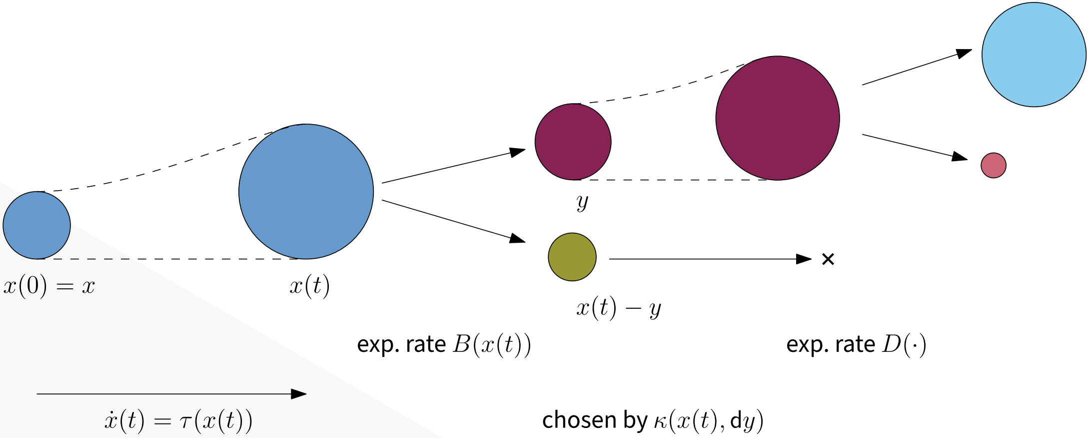
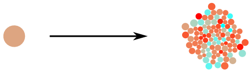


- List sizes at time t: $\mathbf{Z}(t) = (Z_u(t) : u \in U)$

Equilibrium behaviour

t = 0.00
1 cell

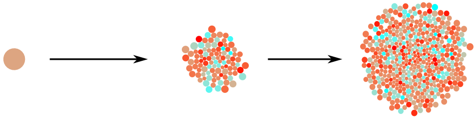
Equilibrium behaviour



t = 0.00
1 cell

t = 4.71
100 cells

Equilibrium behaviour

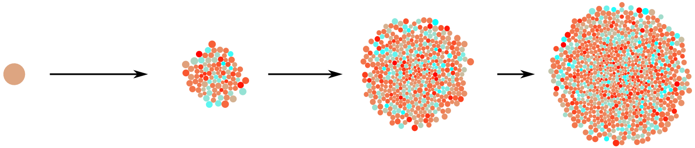


t = 0.00
1 cell

t = 4.71
100 cells

t = 6.32
500 cells

Equilibrium behaviour



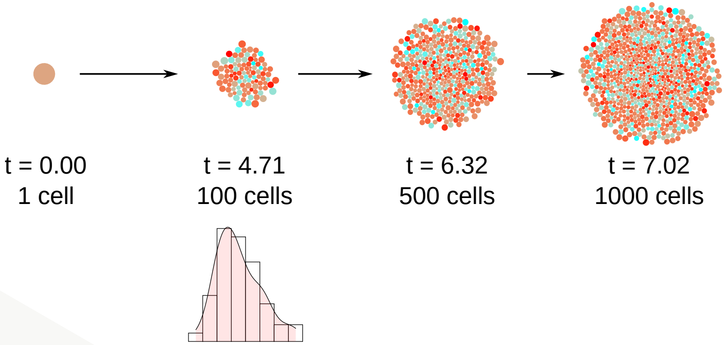
t = 0.00
1 cell

t = 4.71
100 cells

t = 6.32
500 cells

t = 7.02
1000 cells

Equilibrium behaviour



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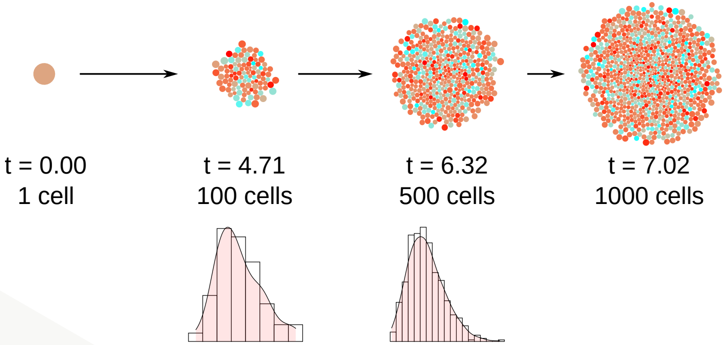

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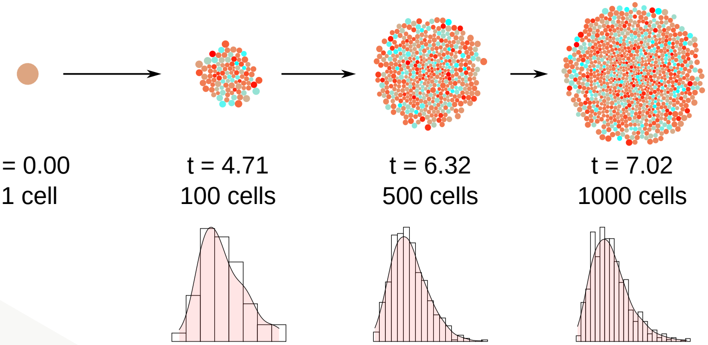

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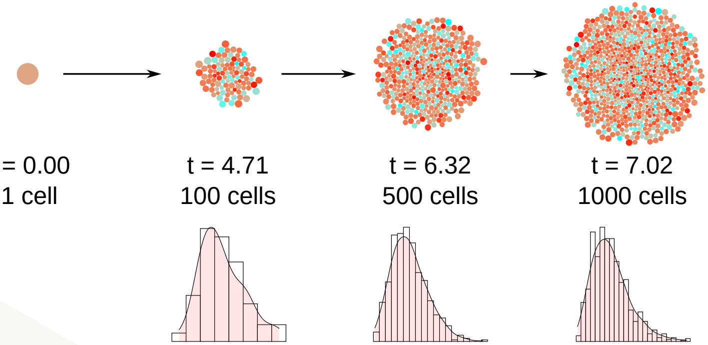

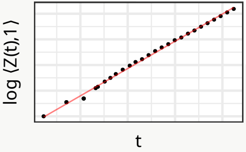

$$\mathbf{Z}(t) \sim Ce^{\lambda t}v$$

Mean measures

Look at $T_t f(x) = \mathbb{E}_x \left[\sum_u f(Z_u(t))\right]$ (**formally**)

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...where $k(x, \mathrm{d}y) = 2B(x)\frac{\kappa(x,\mathrm{d}y)+\kappa(x,x-\mathrm{d}y)}{2}$, and $K(x) = B(x) + D(x)$.

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Questions

- Existence and uniqueness of such T_t? (For which coefficients; for which f?)
- Long term behaviour: $T_t f(x) \sim e^{\lambda t} h(x) \int f(y) \nu(\mathrm{d}y)$? Rate?

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Existing approaches

- Spectral: find $\mathcal{A}h = \lambda h$, $v\mathcal{A} = \lambda v$ and use entropy methods

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- 'Harris-type theorem for non-conservative semigroups': Lyapunov function approach, Bansaye et al. (2019+)

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Heuristic connection with another equation

If $T_t f(x) = \int_0^\infty u_t(x,y)\mathrm{d}y$ and $k(x,\mathrm{d}y) = k(x,y)\mathrm{d}y$, then

$$\partial_t u_t(x,y) + \partial_y(\tau(y) u_t(x,y)) = \int_y^\infty f(z) k(z,y) \mathrm{d}z - K(y) f(y).$$

Our approach

- Try to link to a **killed** Markov process

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- Try to link to a killed Markov process
- Study the quasi-stationary distribution (QSD) (‘stationary after conditioning on survival’)
- Find conditions for existence of the process and its QSD, and link back to desired semigroup T

Existence and uniqueness

Finding a killed Markov process spine

- Fix $\alpha, \beta \in \mathbb{R}$ and let

$$V(x) = \exp\left(-\mathbb{1}_{\{x \le 1\}} \alpha \int_x^1 \frac{\mathrm{d}y}{\tau(y)} + \mathbb{1}_{\{x > 1\}} \beta \int_1^x \frac{\mathrm{d}y}{\tau(y)}\right)$$

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- Let $\mathcal{L}f = \frac{1}{V}\mathcal{A}(fV) - bf$ where $b = \sup_{x>0}\left(\frac{1}{V(x)}\mathcal{A}V(x)\right)$

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- $\mathcal{L}1 \leq 0$; it generates a killed Markov process

Finding a killed Markov process spine

- $\mathcal{L}f(x) = \tau(x)f'(x) + \int_0^x \left[f(y) - f(x)\right] k_V(x, \mathrm{d}y) - q(x)f(x),$

  ($\tau(x)$: growth rate; k_V: jump rate; $q(x)$: killing rate)

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  growth rate (τ), jump rate (k_V), killing rate (q)
- ...where $k_V(x, \mathrm{d}y) = \frac{V(y)}{V(x)} k(x, \mathrm{d}y)$

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  (τ: growth rate; k_V: jump rate; q: killing rate)
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- "$e^{-bt} \frac{1}{V(x)} \mathbb{E}_x \left[\sum_u f(Z_u(t))V(Z_u(t))\right]$" $= e^{-bt} \frac{1}{V(x)} T_t(fV)(x) = \mathbb{E}_x[f(X_t)]$

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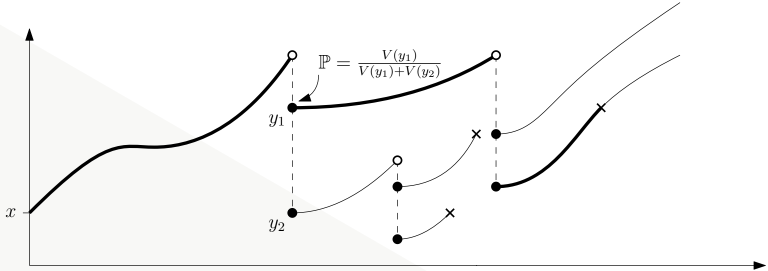

Lemma

Assume, for all $M > 0$,

$$\sup_{x \in (0,M)} k_V(x, (0, x]) < \infty \text{ and } \limsup_{x \to \infty} \left[k_V(x, (0, x]) - K(x) \right] < \infty.$$

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Then there is a Markov process X on $E = (0,\infty)\cup\{\partial\}$ with

$$Q_t f(x) := \mathbb{E}_x[f(X_t)] = f(x) + \int_0^t \mathbb{E}_x[\mathcal{L}f(X_s)]\,\mathrm{d}s$$

$$\mathcal{L}f(x) = \tau(x)f'(x) + \int_0^x \left[f(y) - f(x)\right]k_V(x,\mathrm{d}y) + \left[f(\partial) - f(x)\right]q(x), \quad \mathcal{L}f(\partial) = 0,$$

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for $f : E \to \mathbb{R}$ such that $f|_{(0,\infty)}$ compactly supported and suitably differentiable.
Moreover, Q is the unique semigroup with these properties.

Proof ideas

- Construction: follow the ODE $\dot{x}(t) = \tau(x(t))$, jump at rate k_V, follow ODE from new position...

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- A bit of legwork yields X, unique solution of martingale problem
- Most difficult part: uniqueness of the semigroup
  - Show **any** solution does not approach ∞ or 0 (supermartingale argument)
  - Compare solutions with solutions of martingale problem (a priori not necessarily the same!)

Theorem

Let

$$\mathcal{A}f(x) = \tau(x)f'(x) + \int_0^x f(y)k(x,\mathrm{d}y) - K(x)f(x)$$
$$\mathcal{D}(\mathcal{A}) = \{f\colon (0,\infty) \to \mathbb{R} \text{ suitably differentiable, compactly supported}\} \cup \{V\}.$$

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Then there exists a unique semigroup T such that

$$T_t f(x) = f(x) + \int_0^t T_s \mathcal{A}f(x)\, \mathrm{d}s, \quad f \in \mathcal{D}(\mathcal{A}),$$

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and

$$T_t f(x) = e^{bt} V(x) \mathbb{E}_x[f(X_t)/V(X_t)].$$

‘Unbias the spine motion and add the branching back in’.

Long-term behaviour

Quasi-stationary distributions

- If X is a Markov process killed at T_∂, Champagnat and Villemonais (2018+) give criteria for

$$\mathbb{P}_x(X_t \in \mathrm{d}y \mid T_\partial > t) \to \nu^X(\mathrm{d}y),$$

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- ν^X is the **quasi-stationary distribution**.
- X is **killed** at random rate, our T has **branching** at random rate...

Theorem

In addition to our assumption about k_V, assume that

$$\int_0^\infty \mathbb{1}_{\{k(y,(0,x])>0\}}\,\mathrm{d}y > 0, \quad \text{for } x > 0,$$

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$$k(x,\cdot) \geq \mu, \quad \text{for } x \in I,$$

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and the existence of **Lyapunov functions** ψ, ϕ such that

$$\begin{aligned} \mathcal{A}\psi(x) &\leq \lambda_1 \psi(x) + C\mathbb{1}_L(x), \\ \mathcal{A}\phi(x) &\geq \lambda_2 \phi(x), \end{aligned}$$

with L compact, $\inf \psi/V > 0$, $\sup \phi/V < \infty$, $\inf \tau\phi' > -\infty$, and $\phi(x)/\psi(x) \to 0$ as $x \to 0, \infty$.

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Then...

Theorem

…there exist $\lambda \in \mathbb{R}$, ν a measure, h a function and $\gamma > 0$, such that

$$\left\| e^{-\lambda t} T_t f(x) - h(x) \int f \, \mathrm{d}\nu \right\|_{TV} \leq C e^{-\gamma t} \psi(x)$$

with $T_t h = e^{\lambda t} h$ and $\nu T_t = e^{\lambda t} \nu$.

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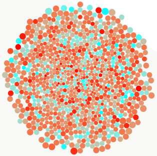

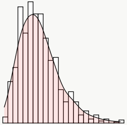

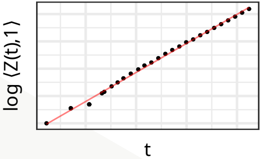

"$\mathbb{E}\mathbf{Z}(t) \sim e^{\lambda t} h(x) \nu$"

OK, BUT CAN YOU ACTUALLY PROVE ANYTHING?

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- ‘A spatially dependent fragmentation process’ – Callegaro and Roberts (2021+)

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- Infer the bias in how the tag is transferred to offspring?

Further reading

D. Villemonais and A. R. Watson
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